Computing with Large Time Steps in Time-Domain Electromagnetics

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The finite Volume Time Domain (FVTD) method is recast in a Large Time Step (LTS) version to solve the time domain Maxwell's equations. LTS allows the use of time steps much larger than that dictated by the Courant Friedrich Lewy (CFL) criteria for numerical stability. Solutions can be obtained much faster and with higher accuracy compared to conventional excplicit time stepping methods.

***Index Terms*—FVTD, Godunov's algorithm, stability criteria, Riemann problem.**

I. Introduction

PRACTICAL electromagnetic (EM) problems often involve scattering from electrically large objects as well as re-entrant structures. Large electrical sizes lead to very fine meshes arising from stringent points-per-wavelength (PPW) requirements to adequately resolve the physical process in a discrete computational framework. The presence of re-entrant structures usually leads to long simulation times due to multiple internal reflections. Full wave solvers in the time domain like Finite Difference Time Domain (FDTD) and Finite Volume Time Domain (FVTD) despite many attractive features, become prohibitively expensive when dealing with such problems. FDTD/FVTD methods are usually based on explicit time marching techniques, and the use of very fine meshes leads to an extremely small time step dictated by the CFL criterion for numerical stability resulting in unrealistically long simulation times. Thus, computationally efficient algorithms for solving the time domain Maxwell's equations that can potentially bypass the CFL stability criterion is of major interest. In the present work the LTS method is used in a FVTD framework to accelerate solution of the time domain Maxwell's equations by either bypassing or at least relaxing the CFL criterion. The LTS method is an extension of Godunov's algorithm and was introduced by LeVeque [1] for solving nonlinear hyperbolic conservation laws. It is based on recognizing the fact that the CFL stability criterion is an artificial constraint arising out of posing a physical problem in a discrete framework. In the classical Godunov's method the Riemann problem is solved at each cell interface at each time step to construct the numerical flux function. The CFL stability criterion can also be interpreted as a means of separating individual Riemann problems at neighbouring cell interfaces from interacting with each other. The LTS method for solving nonlinear hyperbolic problems assumes linearity and allows waves in Riemann problems at individual cell faces to cross over each other without change in speed, strength and creation of any new waves due to interaction. This allows the conventional CFL restriction to be bypassed resulting in large time steps in a numerical time evolution [1]. This approach when applied to purely linear problems like the time-domain Maxwell's equations, can in-principle provide an exact solution to the discrete problem with very large time steps and is equivalent to solving exactly along characteristics. In this work the LTS method is applied to a Godunov based FVTD solution of the time-domain Maxwell's equations. Numerical results are presented for EM field propagation in 1D and 2D in free-space. It is shown that an infinite time step is possible in unbounded domains while a CFL number much greater than one is possible for bounded domains along with superior numerical accuracy compared to conventional time-stepping.

II. Numerical Formulation

Initially the LTS approach is demonstrated for the $x-$ directed and $y-$ polarized 1D TEM waves

$$\frac{\partial E_y}{\partial t} = -\frac{1}{\epsilon}\frac{\partial H_z}{\partial x}, \tag{1a}$$

$$\frac{\partial H_z}{\partial t} = \frac{1}{\mu}\frac{\partial E_y}{\partial x} \tag{1b}$$

where, E_y, H_z respectively are the electric and magnetic fields, ϵ and μ the permittivity and permeability of the medium. For the implementation of the LTS approach in FVTD framework, equation (1) is recast into the conservation form

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{f}(\mathbf{u})}{\partial x} = 0 \tag{2}$$

In the FVTD formulation conservative form of Maxwell's equations are solved in the integral form. The integral form of equation (2), for a finite volume $v_i = (x_{i-1/2}, x_{i+1/2})$ and explicit time step from time t_n to t_{n+1} is written in a discretized cell centered formulation as

$$\mathbf{U}_i^{n+1} = \mathbf{U}_i^n - \frac{\Delta t}{\Delta x}\Big(\mathbf{F}(\mathbf{U}_{i+1/2}^n) - \mathbf{F}(\mathbf{U}_{i-1/2}^n)\Big) \tag{3}$$

where

$$\mathbf{U}_i^n \approx \frac{1}{\Delta x}\int_{v_i} \mathbf{u}(x, t_n)dx \tag{4}$$

is the approximated cell average value and

$$\mathbf{F}(\mathbf{U}_{i+1/2}^n) \approx \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} \mathbf{f}(\mathbf{u}(x_{i+1/2}, t))dt \tag{5}$$

the numerical flux function. This discretized equation (3) is solved using LeVeque LTS approach, which is based on the generalized wave propagation form of Godunov's algorithm. The algorithm in 1D is as follows

Step 1: The spatial domain is subdivided into number of intervals (grid cells) and over each cell conserved variable $\mathbf{u}$ approximated as in (4).

Step 2: Equation (3) is solved using solution of Riemann problem to approximate the numerical flux function at each interface of the grid cell. Hence, the discontinuity at each interface is decomposed into m waves

$$\mathbf{U}_i^n - \mathbf{U}_{i-1}^n = \sum_{p=1}^{m} \Delta \mathbf{u}_{i-1/2}^p = \sum_{p=1}^{m} \alpha_{i-1/2}^p \mathbf{r}^p \quad (6)$$

For the 1D case $m = 2$ with wave speeds $(\lambda^1, \lambda^2) = (-c, c)$. The wave speeds are eigenvalues of Jacobian matrix $\partial \mathbf{f}(\mathbf{u})/\partial x$ with $c = 1/\sqrt{\mu\epsilon}$. $\mathbf{r}^p$ are the eigenvectors of $\partial \mathbf{f}(\mathbf{u})/\partial x$ and $\alpha_{i-1/2}^m$ scalar coefficient of eigenvectors $\mathbf{r}^p$ in the eigenvector expansion (6). If the p^{th} wave propagates through the entire cell interval $(x_{i-1/2}, x_{i+1/2})$ in time Δt, then the p^{th} contribution to the cell average at 'i' is incremented by $\Delta \mathbf{u}_{i-1/2}^p$. The increment is a fraction when the wave propagates a portion of the cell interval. In LTS algorithm, the Riemann problem at each interface is solved independently. Thus, as shown in Fig. 1. the cell

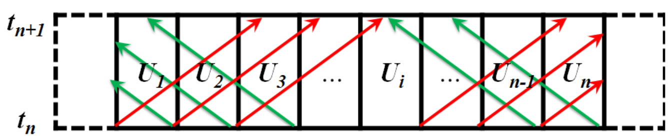


Fig. 1. The schematic of wave propagation in computational domain

averages are updated with the combined contribution of each wave which is cross that interval at time t^{n+1}. For a piecewise constant approximated $\mathbf{u}$ over cell i, the LTS update can be written as

$$\mathbf{U}_i^{n+1} = \mathbf{U}_i^n - \Big(\Delta u_{i-1/2}^2 + \cdots + (\nu - \mu)\Delta u_{i-\mu-1/2}^2\Big)_{\lambda^2 > 0} + \Big(\Delta u_{i+1/2}^1 + \cdots + (\nu - \mu)\Delta u_{i+\mu+1/2}^1\Big)_{\lambda^1 < 0} \quad (7)$$

where, $\nu = |\frac{\lambda_{max}\Delta t}{\Delta x}| \gg 1$ is the Courant or CFL number and μ is it integer counterpart. The superscript 1 and 2 indicates the left and right moving waves in Fig. 1. In conventional time evolution ν is restricted by the CFL stability criterion to a value usually ≤ 1.

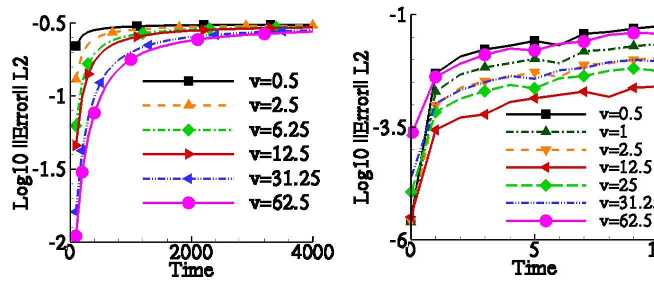
Step 3: Update boundary cells.

The 1D LTS formulation can be formally extended to multidimensions using a dimensional splitting method. This method simplifies the multidimensional problem into a sequence of 1D operators.The CFL number is defined as $\nu = max(|\frac{\lambda_{max}\Delta t}{\Delta x}|, |\frac{\lambda_{max}\Delta t}{\Delta y}|)$ with Δx and Δy cell faces in the Cartesian X and Y directions.

Each 1D sub-problem is considered as an independent self contained computation and boundary cells updated after each dimensional sweep [2].

III. Numerical Results

Results are presented for EM wave propagation in freespace in 1D [3] and 2D [4] involving periodic and perfect conducting (PEC) boundary conditions. The LTS algorithm recovers an exact solution with infinitely large time step in case of 1D EM wave propagation while in case of 2D problems, it allows finite but very large time step. Fig. 2(a) and 2(b) shows logarithmic



(a) 1D EM wave propagation (b) 2D EM wave propagation

Fig. 2. Errors in L_2 norm with varying ν and PEC boundary conditions

error with varying ν as a function of time for both 1D and 2D cases. $\nu \gg 1$ has been used to obtain solution of comparable or superior accuracy to that bounded by the CFL criteria $\nu < 1$.

IV. Conclusion

LTS is implemented for FVTD solution of the time domain Maxwell's equations. The 1D study shows that the LTS scheme retains an exact solution for infinitely large time step. Extension to 2D is through dimensional splitting, the larger ν value again provides higher accuracy due to reduced cumulative numerical errors because of much fewer time steps.

The LTS technique in conjunction with the FVTD method has the potential to be a very effective way to reduce long simulation times encounted in the numerical solution of EM waves involving large electrical sizes and re-entrant structures.

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